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# On some topological invariants related to localized wave functions

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## 1 Introduction

In this note, we consider topological invariants defined for some self-adjoint operators. We are interested in such operators modeled on Hamiltonians which are treated in condensed matter physics. We review mathematical aspects of the *bulk-edge correspondence*, which first appeared in the theoretical study of the quantum Hall effect [15, 8], and also its (mathematical) variants based on the author's work [9, 10].

The bulk-edge correspondence can be understood as a relation between two topological invariants. One is defined for a gapped Hamiltonian on an infinite system without boundary. This is a model of *bulk* (see Fig 1) and this invariant is called the *bulk index.* A model which comes from an insulator will satisfy this gapped condition. The other is defined for a Hamiltonian on a system with codimension-one boundary (*edge*). This invariant is related to wave functions localized near the edge (whose eigenvalues are at the Fermi level of the system) and called the *edge index.* The existence of such localized wave functions means that the edge is metallic. By the bulk-edge correspondence, such wave functions appear reflecting some topology of the bulk. These topological invariants can be treated in the framework of $K$-theory and index theory, and these widely developed theories are applied to condensed matter physics [5, 6, 11, 14].

By combining J. Kellendonk, T. Richter and H. Schulz-Baldes' idea to prove the bulk-edge correspondence [11] and E. Park's result [13], the author defined some secondary invariants compared to the above mentioned invariants, and showed a relation with wave functions on a system with codimension-two boundary (*corner*) [10]. More precisely, we consider a three dimensional system with two edges. As an intersection of these two edges (or as a boundary of codimension-one boundary), our system has a corner (Fig 2). On this system, we consider a Hamiltonian which has a spectral gap not only at the bulk but also at two edges. From the point of view of the bulk-edge correspondence, such Hamiltonian can be regarded as a trivial one since bulk index and edge index are both zero[1]. However, we can still define some secondary topological invariants for such gapped Hamiltonians, and these invariants are related to wave functions (whose eigenvalues are at the Fermi level) on the system with codimension-two corner.

In this note, we review these topological invariants and relations on codimension-one and two boundary systems. In section 3, we consider a two dimensional system with edge (Fig 1), and review a proof of the bulk-edge correspondence from the point of view of $K$-theory and index theory following [9]. We see that the bulk-edge correspondence follows directly from the cobordism invariance of the index, which is a basic property of the index. We here clarify a geometric picture behind this correspondence. In section 4, we consider a system with corner. We here review definitions of above mentioned topological invariants and their relation following [10].

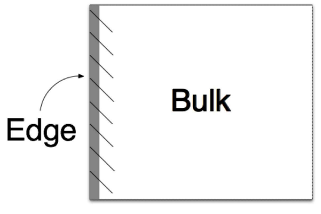


Fig 1: Bulk and edge

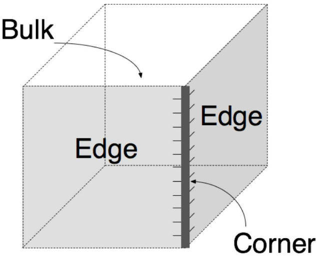


Fig 2: Bulk, edges and corner

## 2 Preliminaries

In this note, let $\mathbb{T} = \mathbb{S}^1_\eta$ be the unit circle in the complex plane[2] and $V$ be a finite rank hermitian vector space whose rank is $N$.

We use topological $K$-theory and $K$-theory for $C^*$-algebras. Our perspective is based on M. F. Atiyah and I. M. Singer's work [2]. We decided not to explain these theories[3] since there are many articles on these topics (e.g. [3, 12]) and the results we use here are briefly summarized in [9, 10]. In order to fix notations, we summarize here the basics of quarter-plane Toeplitz operators which will be used in section 4.

Let $\mathcal{H}$ be a Hilbert space $l^2(\mathbb{Z} \times \mathbb{Z})$. For $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, let $e_{m,n}$ be an element of $\mathcal{H}$ which is 1 at $(m, n)$ and 0 elsewhere. Let $M_{m,n}$ be a translation operator on $\mathcal{H}$ defined by

[1] We here consider what is called a *strong invariant*. A relation with a *weak invariant* and our invariants defined at section 4 is not at all clear.

[2] We use two notations for the same object since two parameters parametrized by these unit circles will play different roles and it will be convenient to distinguish these two parameter spaces.

[3] We also need to clarify our sign convention, but we decided not to explain them neither. On sign convention, we follow the one used in [9, 10].

$(M_{m,n}\varphi)(k,l) = \varphi(m+k, n+l)$. Let $\alpha < \beta$ be real numbers[4]. Let $\mathcal{H}^\alpha$ and $\mathcal{H}^\beta$ be closed subspaces of $\mathcal{H}$ spanned by subsets $\{e_{m,n} \mid -\alpha m + n \geq 0\}$ and $\{e_{m,n} \mid -\beta m + n \leq 0\}$, respectively. Let $P^\alpha$ and $P^\beta$ be orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}^\alpha$ and $\mathcal{H}^\beta$, respectively. Then $P^\alpha P^\beta$ is an orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}^{\alpha,\beta} := \mathcal{H}^\alpha \cap \mathcal{H}^\beta$. Let $\mathcal{T}^{\alpha,\beta}$ be the $C^*$-algebra generated by $\{P^\alpha P^\beta M_{m,n} P^\alpha P^\beta \mid (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$, called the *quarter-plane Toeplitz $C^*$-algebra.* We also define *half-plane Toeplitz $C^*$-algebras* $\mathcal{T}^\alpha$ and $\mathcal{T}^\beta$ to be $C^*$-algebras generated by $\{P^\alpha M_{m,n} P^\alpha \mid (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$ and $\{P^\beta M_{m,n} P^\beta \mid (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$, respectively. We have surjective $*$-homomorphisms $\gamma^\alpha \colon \mathcal{T}^{\alpha,\beta} \to \mathcal{T}^\alpha$, $\gamma^\beta \colon \mathcal{T}^{\alpha,\beta} \to \mathcal{T}^\beta$, $\sigma^\alpha \colon \mathcal{T}^\alpha \to C(\mathbb{T}\times\mathbb{T})$ and $\sigma^\beta \colon \mathcal{T}^\beta \to C(\mathbb{T}\times\mathbb{T})$ , which map $P^\alpha P^\beta M_{m,n} P^\alpha P^\beta$ to $P^\alpha M_{m,n} P^\alpha$, $P^\alpha P^\beta M_{m,n} P^\alpha P^\beta$ to $P^\beta M_{m,n} P^\beta$, $P^\alpha M_{m,n} P^\alpha$ to $\chi_{m,n}$ and $P^\beta M_{m,n} P^\beta$ to $\chi_{m,n}$, respectively, where $\chi_{m,n}$ is a continuous function on $\mathbb{T} \times \mathbb{T}$ defined by $\chi_{m,n}(z_1, z_2) = z_1^m z_2^n$. We define a $C^*$-algebra $\mathcal{S}^{\alpha,\beta}$ to be the pullback of $\mathcal{T}^\alpha$ and $\mathcal{T}^\beta$ along $C(\mathbb{T} \times \mathbb{T})$. There is a surjective $*$-homomorphism $\gamma \colon \mathcal{T}^{\alpha,\beta} \to \mathcal{S}^{\alpha,\beta}$ given by $\gamma(T) = (\gamma^\alpha(T), \gamma^\beta(T))$. Let $\mathcal{K}$ be the $C^*$-algebra of compact operators on $\mathcal{H}^{\alpha,\beta}$. The following will be a key result in section 4.

**Theorem 2.1** (Park [13]). *There is the following short exact sequence for $C^*$-algebras,*

$$0 \to \mathcal{K} \to \mathcal{T}^{\alpha,\beta} \xrightarrow{\gamma} \mathcal{S}^{\alpha,\beta} \to 0,$$

*which has a linear splitting[5] given by a compression onto $\mathcal{H}^{\alpha,\beta}$.*

## 3 Bulk-edge correspondence

In this section, we consider a two-dimensional discrete system without and with edge (codimension-one boundary), and consider the bulk-edge correspondence. We review a proof of this correspondence based on [9]. For a comprehensive account on this topic, see E. Prodan and Schulz-Baldes' book [14].

### 3.1 Topological invariants and the bulk-edge correspondence

Let $A_j \colon \mathbb{T} \to \mathrm{End}_{\mathbb{C}}(V)$ $(j \in \mathbb{Z})$ be continuous maps which satisfies $\sum_{j\in\mathbb{Z}} ||A_j||_\infty < +\infty$, where $||A_j||_\infty = \sup_{t\in\mathbb{T}} ||A_j(t)||_V$ and let $\mu \in \mathbb{R}$. For each $t$ in $\mathbb{T}$, we define a bounded linear map $H(t)$ on the Hilbert space $l^2(\mathbb{Z}; V)$ by $(H(t)\varphi)_n = \sum_{j\in\mathbb{Z}} A_j(t)\varphi_{n-j}$. We assume that $H(t)$ is a self-adjoint operator for any $t$ in $\mathbb{T}$. We also assume that $\mu \notin \mathrm{sp}(H(t))$ for any $t \in \mathbb{T}$, which we call the *spectral gap condition.* Note that $H(t)$ is translation invariant. We call $H(t)$ a *bulk Hamiltonian*[6].

[4]We could take $\alpha = -\infty$ or $\beta = +\infty$ but not both.

[5]This sequence splits as a short exact sequence of linear spaces, and does not split as that of $C^*$-algebras.

[6]An example is given by a partial Fourier transform of a bounded self-adjoint operator on $l^2(\mathbb{Z} \times \mathbb{Z})$ whose spectrum does not contain $\mu$. By the spectral gap condition, we regard $H(t)$ as a model of an insulator.

Let $\gamma$ be a simple closed smooth loop through $\mu$ in $\mathbb{C}$ which surrounds the part of spectrum of $H(t)$ less than $\mu$ for any $t \in \mathbb{T}$ (Fig 3). By the Fourier transform $l^2(\mathbb{Z};V) \cong$

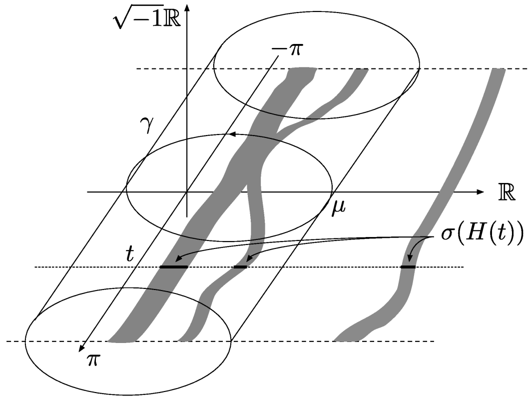


Fig 3: For $t \in \mathbb{T}$, bold black lines indicate the spectrum of the bulk Hamiltonian $\mathrm{sp}(H(t))$. The family of spectrums of bulk Hamiltonians makes a gray area. The loop $\gamma$ (and the torus $\gamma \times \mathbb{T}$) is indicated as a (family of) circle(s)

$L^2(\mathbb{S}^1_\eta;V)$, we have a continuous family of Hermitian endomorphisms on $\mathbb{S}^1_\eta \times \mathbb{T}$ given by $H(\eta,t) = \sum_{j\in\mathbb{Z}} A_j(t)\eta^j \in \mathrm{End}_{\mathbb{C}}(V)$. By the spectral gap condition, Riesz projections $\frac{1}{2\pi i}\int_\gamma (\lambda\cdot 1 - H(\eta,t))^{-1}d\lambda$ give a continuous family of projections on $V$ parametrized by $\mathbb{S}^1_\eta \times \mathbb{T}$. The images of this family makes a complex subvector bundle $E_{\mathrm{B}}$ of the product bundle $(\mathbb{S}^1_\eta \times \mathbb{T}) \times V$. $E_{\mathrm{B}}$ is called the *Bloch bundle*. The Bloch bundle defines a class $[E_{\mathrm{B}}]$ in the $K$-group $K^0(\mathbb{S}^1_\eta \times \mathbb{T})$. We fix a counter-clockwise orientation on $\mathbb{T}$ and $\mathbb{S}^1_\eta$. Each oriented circle has a unique spin$^c$ structure compatible with their fixed orientation up to isomorphism. By using the product spin$^c$ structure on $\mathbb{S}^1_\eta \times \mathbb{T}$, we have a $K$-theoretic push-forward map $\mathrm{ind}_{\mathbb{S}^1_\eta\times\mathbb{T}}\colon K^0(\mathbb{S}^1_\eta \times \mathbb{T}) \to \mathbb{Z}$.

Definition 3.1. We define the *bulk index* of our system by $\mathcal{I}_{\mathrm{Bulk}} := -\,\mathrm{ind}_{\mathbb{S}^1_\eta\times\mathbb{T}}([E_{\mathrm{B}}])$.

Remark 3.2. By using Atiyah-Singer's index formula, it is easy to see that $\mathrm{ind}_{\mathbb{S}^1_\eta\times\mathbb{T}}([E_{\mathrm{B}}])$ is equal to the first Chern number of the Bloch bundle. Our bulk index is equal to the TKNN number [15].

For each $k \in \mathbb{Z}$, let $\mathbb{Z}_{\geq k} := \{k, k+1, k+2, \ldots\}$. Let $P_{\geq k}$ be an orthogonal projection of $l^2(\mathbb{Z};V)$ onto $l^2(\mathbb{Z}_{\geq k};V)$. For each $t$ in $\mathbb{T}$, we consider an operator $H^{\#}(t)$ given by the compression of $H(t)$ onto $l^2(\mathbb{Z}_{\geq 0};V)$, that is, $H^{\#}(t) := P_{\geq 0}H(t)P_{\geq 0}\colon l^2(\mathbb{Z}_{\geq 0};V) \to l^2(\mathbb{Z}_{\geq 0};V)$. We call $H^{\#}(t)$ an *edge Hamiltonian*.

Definition 3.3. We define the *edge index* of our system as the minus of the spectral flow of the family $\{H^{\#}(t)-\mu\}_{t\in\mathbb{T}}$, that is, $\mathcal{I}_{\mathrm{Edge}} := -\mathrm{sf}(\{H^{\#}(t)-\mu\}_{t\in\mathbb{T}})$.

The bulk-edge correspondence for two-dimensional type A topological insulators is the following.

**Theorem 3.4.** *The bulk index equals to the edge index. That is,* $\mathcal{I}_{\mathrm{Bulk}} = \mathcal{I}_{\mathrm{Edge}}$.

This correspondence was first proved by Y. Hatsugai [8], and now many proofs and generalizations are known [11, 7, 14, 9]. In what follows in this section, we present one elementary proof given by the author in [9] and show that the bulk-edge correspondence follows from the cobordism invariance of the index. This result was obtained by studying G. M. Graf and M. Porta's idea [7] from $K$-theoretic point of view and gives a generalization of Graf-Porta's proof. Following [7, 9], we first introduce another vector bundle over $\gamma \times \mathbb{T}$.

## 3.2 Proof of Theorem 3.4

In this section, we define a vector bundle over $\gamma \times \mathbb{T}$, which is a generalization of the one originally used to the proof of the bulk-edge correspondence by Graf and Porta [7].

**Lemma 3.5.** *There exists a positive integer $K$ such that for any integer $k \geq K$ and $(z,t)$ in $\gamma \times \mathbb{T}$, the map $P_{\geq k}(H(t) - z)P_{\geq 0} \colon l^2(\mathbb{Z}_{\geq 0}; V) \to l^2(\mathbb{Z}_{\geq k}; V)$ is surjective.*

*Proof.* The detail of this proof is a little bit complicated [9] but its idea is simple. We present here a rough sketch of the idea. Since surjectivity is an open condition, by some approximation argument, we can assume that the hopping matrices of our Hamiltonian is finitely many, that is, $A_j(t) = 0$ except for finitely many $j \in \mathbb{Z}$. Let $(z,t) \in \gamma \times \mathbb{T}$ and $\psi \in l^2(\mathbb{Z}_{\geq k}; V)$. Under this assumption, we want to find some $\varphi \in l^2(\mathbb{Z}_{\geq 0}; V)$ which satisfies the equation $P_{\geq k}(H(t) - z)\varphi = \psi$. Since $H(t) - z$ is invertible, the map $P_{\geq k}(H(t) - z) \colon l^2(\mathbb{Z}; V) \to l^2(\mathbb{Z}_{\geq k}; V)$ is surjective, and so there is some $\varphi \in l^2(\mathbb{Z}; V)$ which satisfies this equation. Since we assumed that the hopping matrices are finitely many, if we take sufficiently large $k$, the terms $\varphi_{-1}, \varphi_{-2}, \cdots$ does not appear in our equation, and so we can find a solution $\varphi$ of our equation $P_{\geq k}(H(t) - z)\varphi = \psi$ in $l^2(\mathbb{Z}_{\geq 0}; V)$. □

We choose such $k \geq K$. Let $H^\flat(z,t) \colon l^2(\mathbb{Z}_{\geq 0}; V) \to l^2(\mathbb{Z}_{\geq 0}; V)$ be the composite of $P_{\geq k}(H(t) - z)P_{\geq 0}$ and the inclusion $l^2(\mathbb{Z}_{\geq k}; V) \hookrightarrow l^2(\mathbb{Z}_{\geq 0}; V)$.

**Lemma 3.6.** *For any $(z,t) \in \gamma \times \mathbb{T}$, $H^\flat(z,t)$ is a Fredholm operator whose Fredholm index is zero. Moreover, the rank of their kernels are constant with respect to the parameter.*

*Proof.* Let $(z,t) \in \gamma \times \mathbb{T}$. Since $H^\flat(z,t)$ is a finite rank perturbation of the Fredholm operator $H^\#(t) - z$, which can be connected continuously to the self-adjoint operator $H^\#(t) - \mu$, the operator $H^\flat(z,t)$ is Fredholm whose Fredholm index is zero. The rest part follows from Lemma 3.5. □

For $(z,t) \in \gamma \times \mathbb{T}$, we have $\mathrm{Coker} H(z,t) = V^{\oplus k}$ by Lemma 3.5, and we set $(E_{\mathrm{GP}})_{z,t} := \mathrm{Ker}(H(z,t))$. Then kernels $E_{\mathrm{GP}} = \bigsqcup_{(z,t)\in\gamma\times\mathbb{T}} (E_{\mathrm{GP}})_{z,t} \to \gamma \times \mathbb{T}$ is a vector bundle. Cokernels also make a vector bundle $\underline{V}^{\oplus k} := (\gamma \times \mathbb{T}) \times V^{\oplus k}$ which is a product bundle over $\gamma \times \mathbb{T}$. We now fix a counter-clockwise orientation on the loop $\gamma$. By using a spin$^c$ structure on $\gamma$ compatible with this orientation, we have a push-forward map $\mathrm{ind}_{\gamma\times\mathbb{T}} \colon K^0(\gamma \times \mathbb{T}) \to \mathbb{Z}$.

Definition 3.7. $\mathcal{I}_{\mathrm{GP}} := \mathrm{ind}_{\gamma\times\mathbb{T}}([E_{\mathrm{GP}}] - [\underline{V}^{\oplus k}])$.

We call this invariant $\mathcal{I}_{\mathrm{GP}}$ *Graf-Porta's index*. As in the case of bulk index, Graf-Porta's index is the same as the first Chern number of the bundle $E_{\mathrm{GP}}$. We prove Theorem 3.4 by showing the following two relations.

**Proposition 3.8.** (1) $\mathcal{I}_{\mathrm{Bulk}} = -\mathcal{I}_{\mathrm{GP}}$, (2) $-\mathcal{I}_{\mathrm{GP}} = \mathcal{I}_{\mathrm{Edge}}$.

*Proof.* Let $\mathbb{D}^2_\eta$ be the closed unit disk with $\partial\mathbb{D}^2_\eta = \mathbb{S}^1_\eta$, and let $\mathbb{D}^2_z$ be the closed domain of the complex plane with $\partial\mathbb{D}^2_z = \gamma$. Let $X := (\mathbb{D}^2_\eta \times \mathbb{D}^2_z) \setminus (\mathbb{S}^1_\eta \times \gamma) \times \mathbb{T}$, $A := \mathbb{S}^1_\eta \times (\mathbb{D}^2_z \setminus \gamma) \times \mathbb{T}$ and $B = (\mathbb{D}^2_\eta \setminus \mathbb{S}^1_\eta) \times \gamma \times \mathbb{T}$. Then $\partial X = A \sqcup B$.

By using the (Fourier transform of the) bulk Hamiltonian $H(\eta,t)$, we can construct an element $\alpha = [X \times V, X \times V; \rho(H(\eta,t) - z)] \in K^0_{\mathrm{cpt}}(X)$ in a $K$-group with compact supports $K^0_{\mathrm{cpt}}(X)$ of $X$, where $\rho$ is a cut-off function on $X$ which is 0 near $\{0\} \times \gamma \times \mathbb{T}$. Consider the following diagram.

$$\begin{tikzcd}
K^0_{\mathrm{cpt}}(A) \arrow[d, "\beta_z^{-1}"'] & K^0_{\mathrm{cpt}}(X) \arrow[l, "i_A^*"] \arrow[r, "i_B^*"'] & K^0_{\mathrm{cpt}}(B) \arrow[d, "\beta_\eta^{-1}"] \\
K^0(\mathbb{S}^1_\eta \times \mathbb{T}) \arrow[d, "\mathrm{ind}_{\mathbb{S}^1_\eta\times\mathbb{T}}"'] & & K^0(\gamma \times \mathbb{T}) \arrow[d, "\mathrm{ind}_{\gamma\times\mathbb{T}}"] \\
\mathbb{Z} & & \mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
i_A^*(\alpha) \arrow[d, mapsto] & \alpha \arrow[l, mapsto] \arrow[r, mapsto] & i_B^*(\alpha) \arrow[d, mapsto] \\
{[E_{\mathrm{B}}]} \arrow[d, mapsto] & & -[E_{\mathrm{GP}}] + [\underline{V}^{\oplus k}] \arrow[d, mapsto] \\
-\mathcal{I}_{\mathrm{Bulk}} & & -\mathcal{I}_{\mathrm{GP}}
\end{tikzcd}$$

where $i_A$ and $i_B$ are natural inclusions, $\beta_z$ and $\beta_\eta$ are Bott periodicity isomorphisms given by a multiplication of Bott elements. There are some constructions of the inverse of this multiplication map. In [1], Riesz projections are used to construct the inverse, and by using this construction, we have $\beta_z^{-1}(i_A^*(\alpha)) = [E_{\mathrm{B}}]$. In [4], a family index of Toeplitz operators are used, and by using this construction, we see that $\beta_\eta^{-1}(i_B^*(\alpha)) = -[E_{\mathrm{GP}}] + [\underline{V}^{\oplus k}]$.[7] By the cobordism invariance of the index, we have $-\mathcal{I}_{\mathrm{Bulk}} + (-\mathcal{I}_{\mathrm{GP}}) = 0$ and (1) is proved.

By comparing $H^{\#}(t)$ and $H^{\flat}(t)$, it is easy to see that we have a linear map $H^{\#}(t) - z \colon (E_{\mathrm{GP}})_{(z,t)} \to V^{\oplus k}$ for $(z,t) \in \gamma \times \mathbb{T}$. The kernel of this map corresponds to solutions $\varphi \in l^2(\mathbb{Z}_{\geq 0}; V)$ of the eigenequation $H^{\#}(t)\varphi = z\varphi$. Note that the edge index $\mathcal{I}_{\mathrm{Edge}}$ is defined

[7] Actually, we calculated a family index of a family $\{H^{\flat}(t) - z\}_{\gamma\times\mathbb{T}}$ explicitly, which is a finite rank perturbation of the family of Toeplitz operators $\{H^{\#}(t) - z\}_{\gamma\times\mathbb{T}}$. The element of $K$-group does not change under this perturbation.

by counting these solutions (counting the number of crossing points with multiplicity of the spectrum of $H^{\#}(t)$ and $\mu$) with sign. By using a bundle homomorphism $E_{\mathrm{GP}} \to (\gamma \times \mathbb{T}) \times V^{\oplus k}$ given by $H^{\#}(t) - z$ and by using excision, we can localize the index theoretic information of a $K$-class $[E_{\mathrm{GP}}] - [\underline{V}^{\oplus k}]$ near the crossing points and obtain a relation between $\mathcal{I}_{\mathrm{GP}}$ and $\mathcal{I}_{\mathrm{Edge}}$. In this way, (2) can be proved. □

Remark 3.9. Kellendonk–Richter–Schulz-Baldes proved the bulk-edge correspondence based on the six-term exact sequence of $K$-theory for $C^*$-algebras associated to the following Toeplitz extension [11],

$$0 \to K(l^2(\mathbb{N})) \to \mathcal{T} \to C(\mathbb{T}) \to 0,$$

where $\mathcal{T}$ is the Toeplitz algebra. In the next section, we replace this Toeplitz extension by the *quarter-plane Toeplitz extension* obtained by Park [13].

Remark 3.10. The bulk-edge correspondence for two dimensional type AII topological insulators can also be proved by using the cobordism invariance of the index [9]. In this case, we use $KSp$-theory introduced by J. D. Dupont instead of complex $K$-theory.

## 4 Bulk-edge and corner correspondence

In this section, we consider a three dimensional system with corner (codimension-two boundary) which appears as an intersection of two edges (or as a boundary of codimension-one boundary). Actually we consider four systems, a system without edge, two systems with edge (without corner) and a system with corner. We define two topological invariants and show some relation between them. The content of this section is based on [10].

Let $\mathcal{H}_V$ be the Hilbert space $l^2(\mathbb{Z}\times\mathbb{Z}; V) = \mathcal{H}\otimes V$, and let $\mathcal{H}_V^\alpha := \mathcal{H}^\alpha\otimes V$, $\mathcal{H}_V^\beta := \mathcal{H}^\beta\otimes V$, $\mathcal{H}_V^{\alpha,\beta} := \mathcal{H}^{\alpha,\beta}\otimes V$, $P_V^\alpha := P^\alpha \otimes 1$ and $P_V^\beta := P^\beta \otimes 1$. Let $\mathbb{T}\times\mathbb{T}\times\mathbb{T} \to \mathrm{End}_{\mathbb{C}}(V)$ be a continuous map. By the partial Fourier transform, we obtain a continuous family of bounded linear operators $\mathbb{T} \to B(l^2(\mathbb{Z}\times\mathbb{Z}; V))$, $t \mapsto H(t)$. Let $\mu \in \mathbb{R}$. We assume that $H(t)$ is a self-adjoint operator for any $t \in \mathbb{T}$ and call $H(t)$ the *bulk Hamiltonian*. We consider the following operators.

$$H^\alpha(t) := P_V^\alpha H(t) P_V^\alpha \colon \mathcal{H}_V^\alpha \to \mathcal{H}_V^\alpha, \;\; H^\beta(t) := P_V^\beta H(t) P_V^\beta \colon \mathcal{H}_V^\beta \to \mathcal{H}_V^\beta.$$

These operators are half-plane Toeplitz operators, which we call *edge Hamiltonians*. We assume that $\mu \notin \mathrm{sp}(H^\alpha(t))$ nor $\mu \notin \mathrm{sp}(H^\beta(t))$ for any $t \in \mathbb{T}$ (*spectral gap condition*).

The pair $(H^\alpha(t), H^\beta(t))$ is a self-adjoint element of a unital $C^*$-algebra $M_N(\mathcal{S}^{\alpha,\beta}\otimes C(\mathbb{T}))$. By the spectral gap condition, $\mu$ is not contained in its spectrum. Let $h$ be a continuous function on $\mathbb{C} \setminus \{\mathrm{Re}(z) = \mu\}$, which is 0 on $\mathrm{Re}(z) > \mu$ and 1 on $\mathrm{Re}(z) < \mu$. By the continuous functional calculous, we have a projective element $p := h(H^\alpha(t), H^\beta(t))$ in this $C^*$-algebra.

Definition 4.1 ([10]). We denote by $\mathcal{I}_{\mathrm{Bulk-Edge}}$ the element $[p]_0$ in the $K$-group $K_0(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T}))$, and call the *bulk-edge invariant.*

We next consider a family of quarter-plane Toeplitz operators $H^{\alpha,\beta}(t) := P_V^\alpha P_V^\beta H(t) P_V^\alpha P_V^\beta$ on $\mathcal{H}_V^{\alpha,\beta}$, which we call *corner Hamiltonians.* By the spectral gap condition and Theorem 2.1, $\{H^{\alpha,\beta}(t) - \mu\}_{t\in\mathbb{T}}$ is a continuous family of self-adjoint Fredholm operators.

Definition 4.2. The family $\{H^{\alpha,\beta}(t) - \mu\}_{t\in\mathbb{T}}$ gives an element $\mathcal{I}_{\mathrm{Corner}}$ of the $K$-group $K_1(C(\mathbb{T}))$. We call this element $\mathcal{I}_{\mathrm{Corner}}$ the *corner invariant.*

By taking a tensor product of the sequence in Theorem 2.1 and $C(\mathbb{T})$, and consider an associated six-term exact sequence, we have a boundary homomorphism $\delta_0\colon K_0(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T})) \to K_1(\mathcal{K} \otimes C(\mathbb{T}))$. Note that $K_1(\mathcal{K} \otimes C(\mathbb{T}))$ is naturally isomorphic to $K_1(C(\mathbb{T}))$ by the stability of $K$-groups.

**Theorem 4.3** ([10]). *Through the isomorphism $K_1(\mathcal{K} \otimes C(\mathbb{T})) \cong K_1(C(\mathbb{T}))$, the map $\delta_0$ maps the bulk-edge invariant to the corner invariant, that is, $\delta_0(\mathcal{I}_{\mathrm{Bulk-Edge}}) = \mathcal{I}_{\mathrm{Corner}}$.*

*Proof.* This theorem is easily proved by following the construction of the map $\delta_0$ and using a linear splitting of the short exact sequence in Theorem 2.1. □

Remark 4.4. The bulk-edge invariant does not change unless the spectral gap of two edges closes. The above map $\delta_0$ is surjective, but not injective [10].

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